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Existence of Solutions of Generalized Vector Variational Inequalities in Reflexive Banach Spaces

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Abstract. The purpose of this paper is to study the solvability for a class of generalized vector variational inequalities in reflexive Banach spaces. Utilizing the KKM-Fan lemma and the Nadler's result, we prove the solvability results for this class of generalized vector variational inequalities for monotone vector multifuctions. On the other hand, we first introduce the concepts of complete semicontinuity and strong semicontinuity for vector multifunctions. Then we prove the solvability for this class of generalized vector variational inequalities without monotonicity assumption by using these concepts and by applying the Brouwer fixed point theorem. The results in this paper are extension and improvement of the corresponding results in Huang and Fang (2006).

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1. Introduction and Preliminaries

It is well known that vector variational inequalities were initially studied by Giannessi (1980) in the setting of finite dimensional Euclidean spaces. Ever since then they have been widely studied and generalized in infinite dimensional spaces since they cover many diverse disciplines such as partial differential equations, optimal control, optimization, mathematical programming, mechanics, and finance, etc as special cases. The reader is referred to Chen (1989), Chen and Yang (1990), Chen (1992), Giannessi (2000), Giannessi and Maugeri (1995), Huang and Fang (2005), Konnov and Yao (1997), Lai and Yao (1996), Siddiqi, Ansari and Khaliq (1995), Yang (1993), Yang (1997), Yang and Goh (1997), Yu and Yao (1996) and the references therein.

Let X and Y be two real Banach spaces, $K \subset X$ be a nonempty, closed and convex subset, and $C \subset Y$ be a closed, convex and pointed cone with apex at the origin and $\mathrm{int}C \neq \emptyset$ where $\mathrm{int}C$ denotes the interior of C. The cone C is called proper if $C \neq Y$. Recall that C is said to be a closed, convex and pointed cone with apex at the origin if C is closed and the following conditions hold:

(i) $\lambda C \subset C, \quad \forall \lambda > 0$;
(ii) $C + C \subset C$;
(iii) $C \cap (-C) = \{0\}$.

Given a closed, convex and pointed cone C with apex at the origin in Y, we can define relations "$\leqslant_C$" and "$\nleq_C$" as follows:

$$x \leqslant_C y \Leftrightarrow y - x \in C$$

and

$$x \nleq_C y \Leftrightarrow y - x \notin C.$$

Moreover, $a \nleq_{\text{int}C} b$ means $b - a \notin \text{int}C$. Clearly "$\leqslant_C$" is a partial order. In this case $(Y, \leqslant_C)$ is called an ordered Banach space ordered by C. Let $L(X, Y)$ denote the space of all continuous linear maps from X into Y. Let $L_c(X, Y)$ be the subspace of $L(X, Y)$ which consists in all completely continuous linear maps from X into Y. Recall that a mapping $g : X \to Y$ is said to be completely continuous if the weak convergence of x_n to x in X implies the strong convergence of $g(x_n)$ to $g(x)$ in Y.

Now we recall the following concepts and lemmas.

DEFINITION 1.1.

(i) A map $A : K \to L(X, Y)$ is said to be monotone if

$$\langle Ax - Ay, x - y \rangle \geqslant_C 0, \quad \forall x, y \in K,$$

where $a \geqslant_C b$ means $a - b \in C$;

(ii) Let $M : L(X, Y) \to L(X, Y)$ be a mapping. A nonempty compact-valued multifunction $T : K \to 2^{L(X,Y)}$ is said to be monotone with respect to M if for each $x, y \in K$,

$$\langle Mu - Mv, x - y \rangle \geqslant_C 0, \quad \forall u \in T(x), \quad v \in T(y).$$

DEFINITION 1.2. A map $f : K \to Y$ is said to be convex if

$$f(tx + (1 - t)y) \leqslant_C tf(x) + (1 - t)f(y), \quad \forall x, y \in K, \quad t \in [0, 1].$$

LEMMA 1.1. *See Nadler* (1969). *Let* $(X, \|\cdot\|)$ *be a normed vector space and* H *be a Hausdorff metric on the collection* $CB(X)$ *of all nonempty,* *closed and bounded subsets of* X*, induced by a metric* d *in terms of* $d(u, v) = \|u - v\|$*, which is defined by*

$$H(U, V) = \max\left(\sup_{u\in U}\inf_{v\in V}\|u - v\|, \sup_{v\in V}\inf_{u\in U}\|u - v\|\right),$$

for U *and* V *in* $CB(X)$*. If* U *and* V *are compact sets in* X*, then for each* $u \in U$*, there exists* $v \in V$ *such that*

$$\|u - v\| \leqslant H(U, V).$$

DEFINITION 1.3.

(i) A map $A : K \to L(X, Y)$ is said to be υ-hemicontinuous if for any given $x, y \in K$, the mapping $t \to \langle A(x + t(y - x)), y - x\rangle$ is continuous at 0^+;

(ii) A nonempty compact-valued multifunction $T : K \to 2^{L(X,Y)}$ is said to be H-hemicontinuous if for any given $x, y \in K$, the mapping $t \to H(T(x + t(y - x)), T(x))$ is continuous at 0^+ where H is the Hausdorff metric defined on $CB(L(X, Y))$.

Recently, Huang and Fang (2005) considered and studied the solvability for a class of vector variational inequalities in reflexive Banach spaces. They first proved the solvability for this class of vector variational inequalities without monotonicity assumption.

THEOREM 1.1. *See Theorem* 2.1 *in Huang and Fang* (2005)*. Let* K *be a nonempty, bounded, closed and convex subset of a real reflexive Banach space* X *and* Y *be a real Banach space ordered by a closed, convex and pointed cone* C *with apex at the origin and* $\text{int}C \neq \emptyset$*. Suppose that* $T : K \to L_c(X, Y)$ *is a completely continuous map and* $f : K \to Y$ *is a completely continuous and convex map. Then, there exists* $x \in K$ *such that*

$$\langle Tx, y - x\rangle + f(y) - f(x) \nleq_{\text{int}C} 0, \quad \forall y \in K.$$

Second, they proved the solvability for this class of vector variational inequalities with monotone mappings.

THEOREM 1.2. *See Theorem 3.1 in Huang and Fang* (2005)*. Let* K *be a nonempty, bounded, closed and convex subset of a real reflexive Banach space* X *and* Y *be a real Banach space ordered by a closed, convex and pointed cone* C *with apex at the origin and* $\text{int}C \neq \emptyset$*. Suppose that* $T : K \to L_c(X, Y)$ *is a* υ*-hemicontinuous and monotone map, and* $f : K \to Y$ *is a completely continuous and convex map. Then, there exists* $x \in K$ *such that*

$$\langle Tx, y-x\rangle + f(y) - f(x) \not\leq_{\mathrm{int}C} 0, \quad \forall y \in K.$$

Let $T : K \to 2^{L(X,Y)}$ be a vector multifunction. For given maps $A : L(X,Y) \to L(X,Y)$ and $f : K \to Y$, let us consider the following generalized vector variational inequality problem (for short, GVVI): Find $x \in K$ and $u \in T(x)$ such that

$$\langle Au, y-x\rangle + f(y) - f(x) \not\leq_{\mathrm{int}C} 0, \quad \forall y \in K.$$

Motivated and inspired by Huang and Fang (2005), we will study the solvability for the above class of GVVIs in reflexive Banach spaces in this paper. Utilizing the KKM-Fan lemma and the Nadler's result, we establish some solvability results for this class of GVVIs with monotone vector multifunctions. On the other hand, we first introduce the concepts of complete semicontinuity and strong semicontinuity for vector multifunctions. Then we prove the solvability for this class of GVVIs without monotonicity assumption by using these concepts and by applying the Brouwer fixed point theorem. The results presented in this paper are the extension and improvement of the corresponding results in Huang and Fang (2005).

2. Solvability of the GVVI with Monotonicity

In this section, we shall prove the solvability for GVVI with monotone vector multifunctions in reflexive Banach spaces by using the KKM-Fan lemma and the Nadler's result. First we recall some concepts and lemmas.

Let D be a nonempty subset of a topological vector space E. A multivalued map $G : D \to 2^E$ is called a KKM map if for each finite subset $\{x_1, x_2, \ldots, x_n\} \subset D$,

$$\mathrm{conv}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i),$$

where $\mathrm{conv}\{x_1, x_2, \ldots, x_n\}$ denotes the convex hull of $\{x_1, x_2, \ldots, x_n\}$.

LEMMA 2.1. *See Fan* (1961). *Let D be an arbitrary nonempty subset of a Hausdorff topological vector space E. Let the multivalued mapping $G : D \to 2^E$ be a* KKM *map such that $G(x)$ is closed for all $x \in D$ and is compact for at least one $x \in D$. Then*

$$\bigcap_{x \in D} G(x) \neq \emptyset.$$

LEMMA 2.2. *See Chen and Yang* (1990). *Let Y be a real Banach space ordered by a closed, convex and pointed cone C with apex at the origin and* $\text{int}C \neq \emptyset$. *Then, for any $a, b, c \in Y$, the following hold:*

(i) $c \not\leqslant_{\text{int}C} a$ *and* $a \geqslant_C b$ *imply that* $c \not\leqslant_{\text{int}C} b$;
(ii) $c \not\geqslant_{\text{int}C} a$ *and* $a \leqslant_C b$ *imply that* $c \not\geqslant_{\text{int}C} b$.

LEMMA 2.3. *Let K be a nonempty, closed and convex subset of a real Banach space X and Y be a real Banach space ordered by a closed, convex and pointed cone C with apex at the origin and* $\text{int}C \neq \emptyset$. *Let $A : L(X, Y) \to L(X, Y)$ be a continuous map, $T : K \to 2^{L(X,Y)}$ be a nonempty compact-valued multifunction which is H-hemicontinuous and monotone with respect to A, and $f : K \to Y$ be a convex map. Then the following are equivalent:*

(a) *there exist $x_0 \in K$ and $u_0 \in T(x_0)$ such that*

$$\langle Au_0, y - x_0\rangle + f(y) - f(x_0) \not\leqslant_{\text{int}C} 0, \quad \forall y \in K;$$

(b) *there exists $x_0 \in K$ such that*

$$\langle Av, y - x_0\rangle + f(y) - f(x_0) \not\leqslant_{\text{int}C} 0, \quad \forall y \in K, v \in T(y).$$

Proof. Suppose that there exist $x_0 \in K$ and $u_0 \in T(x_0)$ such that

$$\langle Au_0, y - x_0\rangle + f(y) - f(x_0) \not\leqslant_{\text{int}C} 0, \quad \forall y \in K.$$

Since T is monotone with respect to A,

$$\langle Av, y - x_0\rangle + f(y) - f(x_0) \geqslant_C \langle Au_0, y - x_0\rangle + f(y) - f(x_0), \ \forall y \in K, v \in T(y).$$

By Lemma 2.2,

$$\langle Av, y - x_0\rangle + f(y) - f(x_0) \not\leqslant_{\text{int}C} 0, \quad \forall y \in K, v \in T(y).$$

Conversely, suppose that there exists $x_0 \in K$ such that

$$\langle Av, y - x_0\rangle + f(y) - f(x_0) \not\leqslant_{\text{int}C} 0, \quad \forall y \in K, v \in T(y).$$

For any given $y \in K$, we know that $y_t = ty + (1 - t)x_0 \in K$, $\forall t \in (0, 1)$ since K is convex. Replacing y by y_t in the above inequality, one derives for each $v_t \in T(y_t)$

$$\begin{aligned}
&\langle Av_t, y_t - x_0\rangle + f(y_t) - f(x_0) \\
&\quad = \langle Av_t, ty + (1 - t)x_0 - x_0\rangle + f(ty + (1 - t)x_0) - f(x_0) \\
&\quad \leqslant_C \langle Av_t, t(y - x_0)\rangle + tf(y) + (1 - t)f(x_0) - f(x_0) \\
&\quad = t[\langle Av_t, y - x_0\rangle + f(y) - f(x_0)].
\end{aligned}$$

By Lemma 2.2,

$$\langle Av_t, y-x_0\rangle + f(y) - f(x_0) \nleq_{\text{int}C} 0, \quad \forall v_t \in T(y_t), \;\; t \in (0,1). \tag{1}$$

Since $T(y_t)$ and $T(x_0)$ are compact, from Lemma 1.1 it follows that for each fixed $v_t \in T(y_t)$ there exists an $u_t \in T(x_0)$ such that

$$\|v_t - u_t\| \leqslant H(T(y_t), T(x_0)).$$

Since $T(x_0)$ is compact, without loss of generality, we may assume that $u_t \to u_0 \in T(x_0)$ as $t \to 0^+$. Since T is H-hemicontinuous, $H(T(y_t), T(x_0)) \to 0$ as $t \to 0^+$. Thus one has

$$\begin{aligned} \|v_t - u_0\| &\leqslant \|v_t - u_t\| + \|u_t - u_0\| \\ &\leqslant H(T(y_t), T(x_0)) + \|u_t - u_0\| \to 0 \quad \text{as } t \to 0^+. \end{aligned}$$

Note that A is continuous. Therefore letting $t \to 0^+$, we obtain

$$\begin{aligned} \|\langle Av_t, y-x_0\rangle - \langle Au_0, y-x_0\rangle\| &= \|\langle Av_t - Au_0, y-x_0\rangle\| \\ &\leqslant \|Av_t - Au_0\|\,\|y-x_0\| \to 0. \end{aligned}$$

Also by (1) we deduce that $\langle Av_t, y-x_0\rangle + f(y) - f(x_0) \in Y \setminus (-\text{int}C)$. Since $Y\setminus(-\text{int}C)$ is closed, we have that $\langle Au_0, y-x_0\rangle + f(y) - f(x_0) \in Y \setminus (-\text{int}C)$, and so

$$\langle Au_0, y-x_0\rangle + f(y) - f(x_0) \nleq_{\text{int}C} 0, \quad \forall y \in K.$$

This completes the proof. □

THEOREM 2.1. *Let K be a nonempty, bounded closed and convex subset of a real reflexive Banach space X and Y be a real Banach space ordered by a proper closed convex and pointed cone C with apex at the origin and* $\text{int}C \neq \emptyset$*. Let $A: L(X,Y) \to L_c(X,Y)$ be a continuous map, $T: K \to 2^{L(X,Y)}$ be a nonempty compact-valued multifunction which is H-hemicontinuous and monotone with respect to A, and $f: K \to Y$ be a completely continuous and convex map. Then there exist $x \in K$ and $u \in T(x)$ such that*

$$\langle Au, y-x\rangle + f(y) - f(x) \nleq_{\text{int}C} 0, \quad \forall y \in K.$$

Proof. Define two multivalued maps $F, G: K \to 2^K$ as follows:

$$F(y) = \{x \in K : \langle Au, y-x\rangle + f(y) - f(x) \nleq_{\text{int}C} 0 \text{ for some } u \in Tx\}, \quad \forall y \in K$$

and

$$G(y)=\{x\in K:\langle Av, y-x\rangle+f(y)-f(x)\nleq_{\mathrm{int}C}0 \text{ for all } v\in Ty\},\quad \forall y\in K.$$

Then $F(y)$ and $G(y)$ are nonempty since $y\in G(y)\cap F(y)$. We claim that F is a KKM mapping. If this is not true, then there exist a finite set $\{x_1,\ldots,x_n\}\subset K$ and $t_i\geqslant 0, i=1,2,\ldots,n$ with $\sum_{i=1}^n t_i=1$ such that

$$x=\sum_{i=1}^n t_i x_i\notin\bigcup_{i=1}^n F(x_i).$$

Hence for any $u\in T(x)$ one has

$$\langle Au, x_i-x\rangle+f(x_i)-f(x)\leqslant_{\mathrm{int}C}0,\quad i=1,2,\ldots,n.$$

It follows that

$$\begin{aligned}0&=\langle Au, x-x\rangle+f(x)-f(x)\\&\geqslant_C\sum_{i=1}^n t_i\langle Au, x-x_i\rangle+f(x)-\sum_{i=1}^n t_i f(x_i)\\&=\sum_{i=1}^n t_i[\langle Au, x-x_i\rangle+f(x)-f(x_i)]\\&\geqslant_{\mathrm{int}C}0\end{aligned}$$

which leads to a contradiction since C is proper. So F is a KKM mapping. Furthermore we can prove that $F(y)\subset G(y)$ for every $y\in K$. Indeed let $x\in F(y)$. Then for some $u\in T(x)$ one has

$$\langle Au, y-x\rangle+f(y)-f(x)\nleq_{\mathrm{int}C}0.$$

Since T is monotone with respect to A,

$$\langle Av, y-x\rangle+f(y)-f(x)\geqslant_C\langle Au, y-x\rangle+f(y)-f(x),\quad\forall y\in K, v\in Ty.$$

By Lemma 2.2 one has

$$\langle Av, y-x\rangle+f(y)-f(x)\nleq_{\mathrm{int}C}0.$$

Hence $F(y)\subset G(y)$ for each $y\in K$, and so G is also a KKM mapping. Now we claim that for each $y\in K$, $G(y)\subset K$ is closed in the weak topology of X. Indeed suppose $\bar{x}\in\overline{G(y)}^w$, the weak closure of $G(y)$. Since X is reflexive, there is a sequence $\{x_n\}$ in $G(y)$ such that $\{x_n\}$ converges weakly to $\bar{x}\in K$. Then we derive for each $v\in Ty$

$$\langle Av, y-x_n\rangle+f(y)-f(x_n)\nleq_{\mathrm{int}C}0$$

which implies that $\langle Av, y-x_n\rangle + f(y) - f(x_n) \in Y\setminus(-\mathrm{int}C)$. Since Av and f are completely continuous and $Y\setminus(-\mathrm{int}C)$ is closed, so $\{\langle Av, y-x_n\rangle + f(y) - f(x_n)\}$ converges strongly to $\langle Av, y-\bar{x}\rangle + f(y) - f(\bar{x})$ and $\langle Av, y-\bar{x}\rangle + f(y) - f(\bar{x}) \in Y\setminus(-\mathrm{int}C)$. Thus we get

$$\langle Av, y-\bar{x}\rangle + f(y) - f(\bar{x}) \not\leqslant_{\mathrm{int}C} 0,$$

and so $\bar{x}\in G(y)$. This shows that $G(y)$ is weakly closed for each $y\in K$. Since X is reflexive and $K\subset X$ is nonempty, bounded, closed and convex, K is a weakly compact subset of X and so $G(y)$ is also weakly compact. According to Lemma 2.1,

$$\bigcap_{y\in K} G(y)\neq\emptyset.$$

This implies that there exists $x_0\in K$ such that

$$\langle Av, y-x_0\rangle + f(y) - f(x_0) \not\leqslant_{\mathrm{int}C} 0, \quad \forall y\in K, v\in T(y).$$

Therefore by applying Lemma 2.3, we conclude that there exist $x_0\in K$ and $u_0\in T(x_0)$ such that

$$\langle Au_0, y-x_0\rangle + f(y) - f(x_0) \not\leqslant_{\mathrm{int}C} 0, \quad \forall y\in K.$$

This completes the proof. □

REMARK 2.1. In Theorem 2.1, $L_c(X,Y)$ and the complete continuity of f cannot be replaced by $L(X,Y)$ and continuity of f, respectively. Indeed, we can only prove that for each $y\in K$, $G(y)$ is closed in the norm topology of X without convexity of $G(y)$ if A maps $L(X,Y)$ into $L(X,Y)$ and f is continuous. So $G(y)$ need not be weakly compact.

If the boundedness of K is dropped off, then we have the following theorem under certain coercivity condition:

THEOREM 2.2. *Let K be a nonempty, closed and convex subset of a real reflexive Banach space X with $0\in K$ and Y be a real Banach space ordered by a proper closed convex and pointed cone C with apex at the origin and* $\mathrm{int}C\neq\emptyset$. *Let $A: L(X,Y)\to L_c(X,Y)$ be a continuous map, $T: K\to 2^{L(X,Y)}$ be a nonempty compact-valued multifunction which is H-hemicontinuous and monotone with respect to A, and $f: K\to Y$ be a completely continuous and convex map. If there exists some $r>0$ such that*

$$\langle Av, y\rangle + f(y) - f(0) \geqslant_{\mathrm{int}C} 0, \quad \forall v\in T(y), y\in K \text{ with } \|y\|=r, \tag{2}$$

then there exist $x \in K$ *and* $u \in T(x)$ *such that*

$$\langle Au, y - x\rangle + f(y) - f(x) \nleq_{\mathrm{int}C} 0, \quad \forall y \in K.$$

Proof. Let $B_r = \{x \in X : \|x\| \leqslant r\}$. By Theorem 2.1, there exist $x_r \in K \cap B_r$ and $u_r \in T(x_r)$ such that

$$\langle Au_r, y - x_r\rangle + f(y) - f(x_r) \nleq_{\mathrm{int}C} 0, \quad \forall y \in K \cap B_r. \tag{3}$$

Putting $y = 0$ in the above inequality, one has

$$\langle Au_r, x_r\rangle + f(x_r) - f(0) \ngeq_{\mathrm{int}C} 0. \tag{4}$$

Combining (2) with (4), we know that $\|x_r\| < r$. For any $z \in K$, choose $t \in (0, 1)$ enough small such that $(1-t)x_r + tz \in K \cap B_r$. Putting $y = (1-t)x_r + tz$ in (3), one has

$$\langle Au_r, (1-t)x_r + tz - x_r\rangle + f((1-t)x_r + tz) - f(x_r) \nleq_{\mathrm{int}C} 0.$$

Since f is convex,

$$\begin{aligned}&\langle Au_r, (1-t)x_r + tz - x_r\rangle + f((1-t)x_r + tz) - f(x_r)\\ &\quad \leqslant_C t\langle Au_r, z - x_r\rangle + (1-t)f(x_r) + tf(z) - f(x_r)\\ &\quad = t[\langle Au_r, z - x_r\rangle + f(z) - f(x_r)].\end{aligned}$$

By Lemma 2.2,

$$\langle Au_r, z - x_r\rangle + f(z) - f(x_r) \nleq_{\mathrm{int}C} 0, \quad \forall z \in K.$$

This completes the proof. □

By Theorems 2.1 and 2.2, we can obtain the following results:

COROLLARY 2.1. *Let* K *be a nonempty, bounded closed and convex subset of* $X = R^n$ *and* Y *be a real Banach space ordered by a proper closed convex and pointed cone* C *with apex at the origin and* $\mathrm{int}C \neq \emptyset$. *Let* $A : L(R^n, Y) \to L(R^n, Y)$ *be a continuous map,* $T : K \to 2^{L(R^n,Y)}$ *be a nonempty compact-valued multifunction which is* H*-hemicontinuous and monotone with respect to* A, *and* $f : K \to Y$ *be a continuous and convex map. Then there exist* $x \in K$ *and* $u \in T(x)$ *such that*

$$\langle Au, y - x\rangle + f(y) - f(x) \nleq_{\mathrm{int}C} 0, \quad \forall y \in K.$$

COROLLARY 2.2. *Let K be a nonempty, closed and convex subset of $X = R^n$ with $0 \in K$ and Y be a real Banach space ordered by a proper closed convex and pointed cone C with apex at the origin and* $\mathrm{int}C \neq \emptyset$. *Let $A: L(R^n, Y) \to L(R^n, Y)$ be a continuous map, $T: K \to 2^{L(R^n,Y)}$ be a nonempty compact-valued multifunction which is H-hemicontinuous and monotone with respect to A, and $f: K \to Y$ be a continuous and convex map. If there exists some $r > 0$ such that*

$$\langle Av, y\rangle + f(y) - f(0) \geqslant_{\mathrm{int}C} 0, \quad \forall v \in T(y), y \in K \text{ with } \|y\| = r, \tag{5}$$

then there exist $x \in K$ and $u \in T(x)$ such that

$$\langle Au, y - x\rangle + f(y) - f(x) \nleq_{\mathrm{int}C} 0, \quad \forall y \in K.$$

3. Solvability of the GVVI without Monotonicity

In this section, we shall present the solvability of the GVVI without monotonicity in reflexive Banach spaces by using Brouwer fixed point theorem. First we recall some lemmas.

LEMMA 3.1. *See Brouwer* (1912). *Let B be a nonempty, compact and convex subset of a finite dimensional space and $g: B \to B$ be a continuous map. Then there exists $x \in B$ such that $g(x) = x$.*

It can be readily seen that the following lemma holds.

LEMMA 3.2. *See Huang and Fang* (2005). *Let X be a real Banach space, $K \subset X$ be a nonempty, bounded, closed and convex subset, and Y be a real Banach space ordered by a closed, convex and pointed cone C. Then the following conclusions hold:*

(i) *If $T: K \to L_c(X, Y)$ is completely continuous, then for any given $y \in K$, the map $g_y: K \to Y$ defined by $g_y(x) = \langle Tx, y - x\rangle$ is completely continuous;*
(ii) *If $T: K \to L(X, Y)$ is continuous, then for any given $y \in K$, the map $g_y: K \to Y$ defined by $g_y(x) = \langle Tx, y - x\rangle$ is continuous.*

In order to establish the main result in this section, we introduce the following concepts.

DEFINITION 3.1. *Let K be a nonempty, closed and convex subset of a real Banach space X and Y be a real Banach space ordered by a closed, convex and pointed cone C with apex at the origin and* $\mathrm{int}C \neq \emptyset$. *Let $f: K \to Y$ and $A: L(X, Y) \to L(X, Y)$ be two single-valued maps, and $T: K \to 2^{L(X,Y)}$ [resp. $T: K \to L(X, Y)$] be a multivalued [resp. single-valued] map. T is said to be*

(i) *completely semicontinuous with respect to* A *and* f *if for each* $y \in K$,

$$\{x \in K : \langle Au, y - x\rangle + f(y) - f(x) \leqslant_{\mathrm{int}C} 0, \quad \forall u \in T(x)\}$$

$$[\text{resp.} \quad \{x \in K : \langle ATx, y - x\rangle + f(y) - f(x) \leqslant_{\mathrm{int}C} 0\}]$$

is open in K *with respect to the weak topology of* X;

(ii) *strongly semicontinuous with respect to* A *and* f *if for each* $y \in K$,

$$\{x \in K : \langle Au, y - x\rangle + f(y) - f(x) \leqslant_{\mathrm{int}C} 0, \quad \forall u \in T(x)\}$$

$$[\text{resp.} \quad \{x \in K : \langle ATx, y - x\rangle + f(y) - f(x) \leqslant_{\mathrm{int}C} 0\}]$$

is open in K *with respect to the norm topology of* X.

REMARK 3.1.

(i) Let K be a nonempty, bounded, closed and convex subset of a real reflexive Banach space X and Y be a real Banach space ordered by a closed, convex and pointed cone C with apex at the origin and $\mathrm{int}C \neq \emptyset$. Let $f : K \to Y$ be completely continuous, $T : K \to L(X, Y)$ be completely continuous and $A : L(X, Y) \to L_c(X, Y)$ be continuous. Then T is completely semicontinuous with respect to A and f. Indeed, it is easy to see that $AT : K \to L_c(X, Y)$ is completely continuous. Hence, according to Lemma 3.2, for each $y \in K$ the mapping $g : K \to Y$ defined by

$$g_y(x) = \langle ATx, y - x\rangle$$

is completely continuous. Thus as (3) in the proof of Theorem 2.1 in Huang and Fang (2005), the following set

$$N_y = \{x \in K : \langle ATx, y - x\rangle + f(y) - f(x) \leqslant_{\mathrm{int}C} 0\}$$

is open in K with respect to the weak topology of X for every $y \in K$.

(ii) Let K be a nonempty, compact and convex subset of a real Banach space X and Y be a real Banach space ordered by a closed, convex and pointed cone C with apex at the origin and $\mathrm{int}C \neq \emptyset$. Let $f : K \to Y$ be continuous, $T : K \to L(X, Y)$ be continuous and $A : L(X, Y) \to L(X, Y)$ be continuous. Then T is strongly semicontinuous with respect to A and f. Indeed, it is easy to see that $AT : K \to L(X, Y)$ is continuous. Hence according to Lemma 3.2, for each $y \in K$ the mapping $g : K \to Y$ defined by

$$g_y(x) = \langle ATx, y - x\rangle$$

is continuous. Thus as in the proof of Theorem 2.2 in Huang and Fang (2005), the following set

$$N_y = \{x \in K : \langle ATx, y-x\rangle + f(y) - f(x) \leqslant_{\mathrm{int}C} 0\}$$

is open in K with respect to the norm topology of X for every $y \in K$.

Next we state and prove the main result in this section.

THEOREM 3.1. *Let K be a nonempty, bounded closed and convex subset of a real reflexive Banach space X and Y be a real Banach space ordered by a proper closed convex and pointed cone C with apex at the origin and* $\mathrm{int}C \neq \emptyset$. *Let $f: K \to Y$ and $A: L(X,Y) \to L(X,Y)$ be two maps such that f is convex, and let $T: K \to 2^{L(X,Y)}$ take nonempty values. If T is completely semicontinuous with respect to A and f, then there exist $x \in K$ and $u \in T(x)$ such that*

$$\langle Au, y-x\rangle + f(y) - f(x) \not\leqslant_{\mathrm{int}C} 0, \quad \forall y \in K.$$

Proof. Suppose that the conclusion is not true. Then for each $x_0 \in K$ there exists some $y \in K$ such that

$$\langle Au_0, y-x_0\rangle + f(y) - f(x_0) \leqslant_{\mathrm{int}C} 0, \quad \forall u_0 \in T(x_0). \tag{6}$$

For every $y \in K$, define the set N_y as follows:

$$N_y = \{x \in K : \langle Au, y-x\rangle + f(y) - f(x) \leqslant_{\mathrm{int}C} 0, \quad \forall u \in T(x)\}. \tag{7}$$

Since T is completely semicontinuous with respect to A and f, the set N_y is open in K with respect to the weak topology of X for every $y \in K$.

Now we assert that $\{N_y : y \in K\}$ is an open cover of K with respect to the weak topology of X. Indeed, first it is easy to see that

$$\bigcup_{y\in K} N_y \subset K.$$

Second, for each $x_0 \in K$, by (6) there exists $y \in K$ such that $x_0 \in N_y$. Hence $x_0 \in \bigcup_{y\in K} N_y$. This shows that $K \subset \bigcup_{y\in K} N_y$. Consequently,

$$K = \bigcup_{y\in K} N_y.$$

So the assertion is valid.

The weak compactness of K implies that there exists a finite set $\{y_1, \dots, y_n\} \subset K$ such that

$$K = \bigcup_{i=1}^{n} N_{y_i}.$$

Hence there exists a continuous (with respect to the weak topology of X) partition of unity $\{\beta_1, \dots, \beta_n\}$ subordinated to $\{N_{y_1}, \dots, N_{y_n}\}$ such that $\beta_j(x) \geqslant 0, \forall x \in K, j = 1, \dots, n,$

$$\sum_{j=1}^{n} \beta_j(x) = 1, \quad \forall x \in K,$$

and

$$\beta_j(x) \begin{cases} = 0, & \text{whenever } x \notin N_{y_j}, \\ > 0, & \text{whenever } x \in N_{y_j}. \end{cases}$$

Let $p : K \to X$ be defined as follows:

$$p(x) = \sum_{j=1}^{n} \beta_j(x) y_j, \quad \forall x \in K. \tag{8}$$

Since β_i is continuous with respect to the weak topology of X for each i, p is continuous with respect to the weak topology of X. Let $S = \text{conv}\{y_1, \dots, y_n\} \subset K$. Then S is a simplex of a finite dimensional space and p maps S into S. By Brouwer fixed point theorem (Lemma 3.1), there exists some $x_0 \in S$ such that $p(x_0) = x_0$. Now for any given $x \in K$, let

$$k(x) = \{j : x \in N_{y_j}\} = \{j : \beta_j(x) > 0\}.$$

Obviously, $k(x) \neq \emptyset$.

Since $x_0 \in S \subset K$ is a fixed point of p, we have $p(x_0) = \sum_{j=1}^{n} \beta_j(x_0) y_j$ and hence from (7) and the convexity of f we derive for each $u_0 \in T(x_0)$

$$\begin{aligned}
0 &= \langle Au_0, x_0 - x_0 \rangle + f(x_0) - f(x_0) \\
&= \langle Au_0, x_0 - p(x_0) \rangle + f(x_0) - f(p(x_0)) \\
&= \left\langle Au_0, x_0 - \sum_{j=1}^{n} \beta_j(x_0) y_j \right\rangle + f(x_0) - f\left(\sum_{j=1}^{n} \beta_j(x_0) y_j \right) \\
&\geqslant_C \sum_{j=1}^{n} \beta_j(x_0)[\langle Au_0, x_0 - y_j \rangle + f(x_0) - f(y_j)] \\
&= \sum_{j \in k(x_0)} \beta_j(x_0)[\langle Au_0, x_0 - y_j \rangle + f(x_0) - f(y_j)] \\
&\geqslant_{\text{int}C} 0
\end{aligned}$$

which leads to a contradiction. Therefore there exist $x \in K$ and $u \in T(x)$ such that

$$\langle Au, y - x \rangle + f(y) - f(x) \not\leqslant_{\text{int}C} 0, \quad \forall y \in K.$$

This completes the proof. □

THEOREM 3.2. *Let K be a nonempty, compact and convex subset of a real Banach space X and Y be a real Banach space ordered by a proper closed convex and pointed cone C with apex at the origin and* $\text{int}C \neq \emptyset$. *Let $f: K \to Y$ and $A: L(X,Y) \to L(X,Y)$ be two maps such that f is convex, and let $T: K \to 2^{L(X,Y)}$ take nonempty values. If T is strongly semicontinuous with respect to A and f, then there exist $x \in K$ and $u \in T(x)$ such that*

$$\langle Au, y - x \rangle + f(y) - f(x) \not\leqslant_{\text{int}C} 0, \quad \forall y \in K.$$

Proof. The proof is similar to that of Theorem 2.1 and so is omitted. □

If $X = R^n$, then $L_c(R^n, Y) = L(R^n, Y)$, complete continuity is equivalent to continuity and complete semicontinuity is equivalent to strong semicontinuity. By Theorem 3.1, we can obtain the following result:

COROLLARY 3.1. *Let K be a nonempty, bounded, closed and convex subset of R^n and Y be a real Banach space ordered by a proper closed convex and pointed cone C with apex at the origin and* $\text{int}C \neq \emptyset$. *Let $f: K \to Y$ and $A: L(R^n, Y) \to L(R^n, Y)$ be two maps such that f is convex, and let $T: K \to 2^{L(R^n,Y)}$ take nonempty values. If T is strongly semicontinuous with respect to A and f, then there exist $x \in K$ and $u \in T(x)$ such that*

$$\langle Au, y - x \rangle + f(y) - f(x) \not\leqslant_{\text{int}C} 0, \quad \forall y \in K.$$

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